Mideterm Exam 2016 , Mathematical Physics, Total number of points 100
10 points free for comming to the exam. For all problems justify your answer.

Problem 1 (20 points)

Prove that $$\lim_{n\to+\infty}\left(1+\frac{3}{n}\right)^{5n}=e^{15}$$

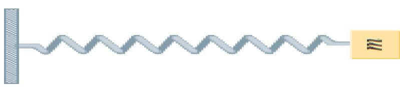
Problem 2 (20 points)

Determine weather the series $\sum_{n=1}^{+\infty}\frac{\cos(4n)}{1+5^n}$ is convergent or divergent.

Problem 3 (20 points) Consider the series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n 4^n}$

(10 points) For which x is the series absolute convergent ?
(5 points) For which x is conditionally convergent ?
(5 points) For which x is divergent?

Problem 4 (30 points)



consider a spring with mass m, spring constant k, and damping constant $c = 0$, and let $\omega = \sqrt{k/m}$. If an external force $F(t) = F_0 \cos \omega t$ is applied (the applied frequency equals the natural frequency), use the method of undetermined coefficients to show that the motion of the mass is given by

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{2m\omega} t \sin \omega t$$

Consider the equation of motion $m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = F(t)$ and set c=0

Problem 1 (20 points)

Set $y = \ln\left[1 + \frac{3}{n}\right]^{5n} = 5n \ln\left[1 + \frac{3}{n}\right]$

$\Rightarrow y = \dfrac{\ln\left[1 + \frac{3}{n}\right]}{\frac{1}{5n}}$ take limit and apply L'Hopital's rule

$$\lim_{n\to\infty} y = \lim_{n\to\infty} \frac{\frac{d}{dn}\ln\left[1 + \frac{3}{n}\right]}{\frac{d}{dn}\left[\frac{1}{5n}\right]} = \lim_{n\to\infty} \frac{-3n^{-2}\,\frac{1}{1+\frac{3}{n}}}{\frac{-n^{-2}}{5}}$$

$$\lim_{n \to \infty} y = 15 \lim_{n \to \infty} \frac{1}{1 + \frac{3}{n}} = 15$$

Thus $\lim_{n \to \infty} \ln\left[1 + \frac{3}{n}\right]^{5n} = 15 \Rightarrow$

$$\boxed{\lim_{n \to \infty} \left(1 + \frac{3}{n}\right)^{5n} = e^{15}}$$

Problem 2 (20 points)

Take $\left|\frac{\cos(4n)}{1+5^n}\right|$ and consider $|\cos(4n)| \leq 1$

$$\left|\frac{\cos(4n)}{1+5^n}\right| \leq \frac{1}{1+5^n} < \frac{1}{5^n}$$

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^{n-1}\left(\frac{1}{5}\right)$$ convergent geometric series with $r = 1/5 < 1$

Thus from the comparison test

$\sum_{n=1}^{\infty}\left|\frac{\cos(4n)}{1+5^n}\right|$ is absolute convergent $\Rightarrow$

$\sum_{n=1}^{\infty} \frac{\cos(4n)}{1+5^n}$ convergent!

Problem 3 (20 points)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x+2|^{n+1}}{(n+1)\,4^{n+1}}\cdot\frac{n\,4^n}{|x+2|^n}\right] = \lim_{n\to\infty}\left[\frac{n}{n+1}\frac{|x+2|}{4}\right] = \frac{|x+2|}{4} < 1 \quad\Leftrightarrow\quad |x+2| < 4,$$

$$|x+2| < 4 \quad\Leftrightarrow\quad -4 < x+2 < 4 \quad\Leftrightarrow\quad -6 < x < 2.$$

(a) Absolute convergent for -6<x<2

If $x = -6$, then the series $\sum_{n=1}^{\infty}\frac{(x+2)^n}{n\,4^n}$ becomes

$\sum_{n=1}^{\infty}\frac{(-4)^n}{n4^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, the alternating harmonic series, which converges by the Alternating Series Test.

(b) Conditionally convergent for x=-6

When $x = 2$, the series becomes the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges.

(c) Diverges for x<-6 and x≥2

Problem 4 (30 points)

Forced Vibrations

See in 17.3, the equation of motion : $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$

And set c=0

The auxiliary equation for the homogenous equation has two imaginary roots $\pm j\omega$ so the solution is:

$$x_c(t)=c_1\cos\omega t+c_2\sin\omega t$$

But the natural frequency of the system equals the

frequency of the external force, so try $x_p(t)=t(A\cos\omega t+B\sin\omega t)$. Then we need

$m(2\omega B-\omega^2 At)\cos\omega t-m(2\omega A+\omega^2 Bt)\sin\omega t+kAt\cos\omega t+kBt\sin\omega t=F_0\cos\omega t$ or $2m\omega B=F_0$ and $-2m\omega A=0$ (noting $-m\omega^2 A+kA=0$ and $-m\omega^2 B+kB=0$ since $\omega^2=k/m$). Hence the general solution is

$$x(t)=c_1\cos\omega t+c_2\sin\omega t+\left[F_0 t/(2m\omega)\right]\sin\omega t$$